

Newton's method in n dimensions

1. Find an approximation of a root of this system of two non-linear equations $\mathbf{f}(\mathbf{u}) = \mathbf{0}$ starting with the given initial vector and performing three iterations of Newton's method in two dimensions:

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} u_1 u_2 + 3u_1 - 1 \\ u_1 + 2u_2 - 1 \end{pmatrix} \text{ and } \mathbf{u}_0 = \begin{pmatrix} 0.3 \\ 0.3 \end{pmatrix}$$

Answer:

```
0.3    0.298412698412698    0.298437881184535    0.298437881283576
0.3    0.350793650793651    0.350781059407733    0.350781059358212
```

2. In all likelihood, you did not do this by hand. How would you author functions in Matlab to calculate both the function and the Jacobian and then to perform the given iteration?

Answer:

```
>> f = @(u)( [u(1)*u(2) + 3*u(1) - 1; u(1) + 2*u(2) - 1] );
>> J = @(u)( [u(2) + 3, u(1); 1 2] );
>> u = [0.3 0.3]';
>> for i = 1:5
    du = J(u) \ -f(u);
    norm( du )
    u = u + du
    f( u )
end
```

3. What is the purpose of printing out the norm of the calculated solution du in Question 2?

Answer: This is identical to calculating $\|\mathbf{u}_{k+1} - \mathbf{u}_k\|_2$.

4. What is $\mathbf{f}(\mathbf{u}_0)$ in Question 1?

Answer:
$$\mathbf{f}(\mathbf{u}_0) = \begin{pmatrix} 0.09 + 0.9 - 1 \\ 0.3 + 0.6 - 1 \end{pmatrix} = \begin{pmatrix} -0.01 \\ -0.1 \end{pmatrix}$$

5. What is $\mathbf{f}(\mathbf{u}_3)$ in Question 1?

Answer:
$$\mathbf{f}(\mathbf{u}_3) = \begin{pmatrix} 0.09 + 0.9 - 1 \\ 0.3 + 0.6 - 1 \end{pmatrix} = \begin{pmatrix} -0.01 \\ -0.1 \end{pmatrix}$$

6. Find an approximation of a root of this system of three non-linear equations $\mathbf{f}(\mathbf{u}) = \mathbf{0}$ starting with the given initial vector and performing four iterations of Newton's method in two dimensions:

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} 5u_1^2 - 2u_1u_2 + u_3 - 3 \\ u_1 - 4u_2^2 - u_1u_3 + 4 \\ u_1u_2 - 3u_3^2 + 3 \end{pmatrix} \text{ and } \mathbf{u}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Answer: To twelve digits after the decimal point,

```
1      0.85323383085 0.84003675797 0.83989349572 0.83989348081 0.83989348081
1      0.98258706468 0.98630689451 0.98629370484 0.98629370416 0.98629370416
1      1.13930348259 1.12972831372 1.12965801322 1.12965800906 1.12965800906
```

7. In all likelihood, you did not do this by hand. How would you author functions in Matlab to calculate both the function and the Jacobian and then to perform the given iteration?

Answer:

```
f = @(u)( [5*u(1)^2 - 2*u(1)*u(2) + u(3) - 3
          u(1) - 4*u(2)^2 - u(1)*u(3) + 4
          u(1)*u(2) - 3*u(3)^2 + 3] );
J = @(u)( [10*u(1) - 2*u(2), -2*u(1), 1
          1 - u(3), -8*u(2), -u(1)
          u(2), u(1), -6*u(3)] );
u = [1 1 1]';
for i = 1:5
    du = J(u) \ -f(u);
    norm( du )
    u = u + du
end
```

8. What is $\mathbf{f}(\mathbf{u}_0)$ in Question 6?

$$\text{Answer: } \mathbf{f}(\mathbf{u}_0) = \begin{pmatrix} 5 - 2 + 1 - 3 \\ 1 - 4 - 1 + 4 \\ 1 - 3 + 3 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$$

9. What is $\mathbf{f}(\mathbf{u}_4)$ in Question 6?

$$\text{Answer: To five significant digits: } \mathbf{f}(\mathbf{u}_4) = \begin{pmatrix} 1.3323 \times 10^{-15} \\ 8.8818 \times 10^{-16} \\ -8.8818 \times 10^{-16} \end{pmatrix}$$

10. Find an approximation of a root of this system of four non-linear equations $\mathbf{f}(\mathbf{u}) = \mathbf{0}$ starting with the given initial vector and performing three iterations of Newton's method in two dimensions:

$$\mathbf{f}(\mathbf{u}) = \begin{pmatrix} \sin(u_1 + u_3 - 5) - 0.3 \\ \sin(u_2 + u_4 + 4) + 0.1 \\ \sin(u_3 + u_2 - 2) - 0.2 \\ \sin(u_1 - u_4 + 3) + 0.4 \end{pmatrix} \text{ and } \mathbf{u}_0 = \begin{pmatrix} 1.0 \\ 1.5 \\ 0.5 \\ 0.5 \end{pmatrix}$$

Answer:

1	1.046821751722210	1.044242520673772	1.044242078639907
1.5	1.692592625721158	1.691885520197611	1.691885307035425
0.5	0.507407374278842	0.509472213281955	0.509472613754902
0.5	0.494265373004080	0.491133087706175	0.491132578982627

11. In all likelihood, you did not do this by hand. How would you author functions in Matlab to calculate both the function and the Jacobian and then to perform the given iteration?

Answer:

```
f = @(u) ( [sin(u(1) + u(3) - 5) - 0.3
           sin(u(2) + u(4) + 4) + 0.1
           sin(u(3) + u(2) - 2) - 0.2
           sin(u(1) - u(4) + 3) + 0.4] );
J = @(u) ( [cos(u(1)+u(3)-5),      0, cos(u(1)+u(3)-5),      0
           0, cos(u(2)+u(4)+4),    0, cos(u(2)+u(4)+4)
           0, cos(u(3)+u(2)-2),    cos(u(3)+u(2)-2),    0
           cos(u(1)-u(4)+3),      0,      0, -cos(u(1)-u(4)+3)] );
u = [1 1.5 0.5 0.5]';
for i = 1:3
    du = J(u) \ -f(u);
    norm( du )
    u = u + du
end
```

12. What is $\mathbf{f}(\mathbf{u}_0)$ in Question 10?

$$\text{Answer: } \mathbf{f}(\mathbf{u}_0) = \begin{pmatrix} \sin(-3.5) - 0.3 \\ \sin(6) + 0.1 \\ \sin(0) - 0.2 \\ \sin(3.5) + 0.4 \end{pmatrix} = \begin{pmatrix} 0.050783 \\ -0.17942 \\ -0.2 \\ 0.049217 \end{pmatrix}$$

13. What is $\mathbf{f}(\mathbf{u}_3)$ in Question 10?

$$\text{Answer: } \mathbf{f}(\mathbf{u}_3) = \begin{pmatrix} -4.4409 \times 10^{-16} \\ 2.5438 \times 10^{-14} \\ -3.4139 \times 10^{-15} \\ 1.1102 \times 10^{-15} \end{pmatrix}$$